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Domain Embedding Preconditioners For Mixed Systems

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In this paper we study block diagonal preconditioners for mixed systems derived from the Dirichlet problems for second order elliptic equations. The main purpose is to discuss how an embedding of the original computational domain into a simpler extended domain can be utilized in this case. We show that a family of uniform preconditioners for the corresponding problem on the extended, or fictitious, domain leads directly to uniform preconditioners for the original problem. This is in contrast to the situation for the standard finite element method, where the domain embedding approach for the Dirichlet problem is less obvious.

KEY WORDS second order elliptic problems; Dirichlet boundary conditions; mixed finite elements; preconditioning; domain embedding; auxiliary space method; non-conforming finite elements

1. Introduction

We consider mixed finite element approximations for second order elliptic equations, with Dirichlet boundary conditions, of the form

$$\begin{aligned} -\mathrm{div}(a\,\mathbf{grad}\,p) &= f \qquad \text{in } \Omega \\ p &= g \qquad \text{on } \partial\,\Omega \end{aligned} \tag{1.1}$$

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Here $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain and $\partial\Omega$ is the boundary. The coefficient matrix a is assumed to be bounded and uniformly positive definite on Ω. The main purpose of this paper is to discuss how to construct preconditioners for the corresponding discrete linear systems where an embedding of the domain Ω into an extended, or fictitious, more regular domain Ω_e is utilized.

If the boundary value problem (1.1) is discretized by a standard conforming finite element method a symmetric and positive definite system of the form

$$A_h p_h = f_h$$

is obtained. Here, $h > 0$ is a small parameter indicating the mesh size. However, since the condition number of the coefficient operator A_h increases with decreasing values of h, the behaviour of iterative methods depends on the construction of effective preconditioners B_h.

In a domain embedding approach we utilize an embedding of the original domain Ω into an extended domain Ω_e to construct a proper preconditioner B_h. This is in contrast to domain decomposition methods where the domain is split into a number of subdomains. In practical applications the geometry of Ω_e will be simpler, or more regular, than the geometry of Ω. Hence, it is reasonable to assume that corresponding preconditioners on the domain Ω_e, for example, constructed by multilevel methods, are more easily obtained. In certain applications, it can even be possible to apply a fast solver on the extended domain. For early papers discussing domain embedding as a tool for solving discrete systems we refer to Astrakhantsev [4] and Buzbee *et al.* [10]. For some more recent developments, cf. for example Proskurowski and Vassilevski [27,28].

For problems of the form (1.1), but with the Dirichlet boundary conditions replaced by natural boundary conditions, application of domain embedding is rather straightforward. In this case we can apply preconditioners B_h of the form

$$B_h = R_h B_{e,h} E_h \tag{1.2}$$

where $B_{e,h}$ is a corresponding preconditioner with respect to the domain Ω_e, E_h is essentially the extension by zero operator and R_h is the operator defined by restricting functions to Ω. Under proper conditions on $B_{e,h}$ the preconditioner B_h will be a uniform preconditioner for A_h, i.e., the spectral condition number of $B_h A_h$ is independent of the mesh parameter h (cf. for example, Astrakhantsev [4] or Marchuk, Kuznetsov and Matsokin [23]).

For Dirichlet (or essential) boundary conditions the situation is not as straightforward. One way to remove the essential boundary condition is to use the Lagrange multiplier method for the Dirichlet problem introduced by Babuška [5]. The resulting saddle point system can then be preconditioned by a block diagonal operator, where one of the blocks will correspond to a preconditioner for the Neumann problem. Hence, this part of the operator can be constructed by domain embedding as indicated above. The second block of the preconditioner is a boundary operator defined on the interface between Ω and $\Omega_+ \equiv \Omega_e \setminus \overline{\Omega}$. For a general description and various applications of Lagrange multipliers and domain embedding for the Dirichlet problem we refer to Dinh *et al.* [13] and Rossi [30]. An alternative approach, utilizing an approximate harmonic extension, is suggested by Nepomnyaschikh [24] and studied in detail in [25,26], while Finogenov and Kuznetsov [14] study a two–stage method for the Dirichlet problem. The latter approach utilizes inexact but sufficiently accurate solutions of problems on the extended domain Ω_e. The problem of inexact solutions on the extended domain was also treated in Börgers and Widlund [6]. Another version of the domain embedding method in the framework of [24], based on recently proposed bounded extension operators exploiting $H^{\frac{1}{2}}$–stable wavelet–like hierarchical decomposition of conforming finite element spaces, was presented in Vassilevski [33].

In the present paper we shall consider the discrete systems obtained from the mixed finite element method for the problem (1.1). This discretization procedure will lead to an indefinite saddle point system. Also, the Dirichlet boundary conditions are natural boundary conditions for the mixed weak formulation of the problem (1.1). Therefore, as we shall see below, domain embedding preconditioners for these systems can be constructed rather naturally of the form (1.2).

In Section 2 we will give a brief review of the mixed finite element method for elliptic boundary value problems of the form (1.1), while preconditioning of saddle point problems is discussed in Section 3. In particular, we will describe the tight connection between preconditioners for the indefinite mixed system and preconditioners for the positive definite operator derived from the inner product in the space $\boldsymbol{H}(\mathrm{div})$. Domain embedding preconditioners for this positive definite operator will be discussed in Section 4. A key tool in the analysis is the construction of an extension operator which is bounded in $\boldsymbol{H}(\mathrm{div})$. In Section 5 we show how the $\boldsymbol{H}(\mathrm{div})$ preconditioner proposed in Section 4 can be combined with the auxiliary space technique (cf. Xu [37]), to construct preconditioners for the systems obtained from the non-conforming Crouzeix–Raviart method by viewing this method as a non-conforming mixed method. Finally, some numerical experiments are presented in Section 6.

We adopt, throughout this paper the following notation: functions and spaces in bold face will denote vector fields and vector valued function spaces, respectively. Similarly, operators in bold face (such as **grad** or **curl**) will have actions in vector-valued function spaces.

2. Preliminaries

The inner product in $L^2(\Omega)$ will be denoted $(\cdot,\cdot)$ and the same notation will be used to denote the corresponding inner product for vector-valued functions. The Sobolev space of functions with derivatives of order less than or equal to m in $L^2(\Omega)$ will be denoted $H^m(\Omega)$, and the associated norm by $||\cdot||_m$. Furthermore, $\boldsymbol{H}(\mathrm{div};\Omega)$ is the space consisting of square integrable 2–vectorfields with square integrable divergence. The inner product in $\boldsymbol{H}(\mathrm{div};\Omega)$ is given by

$$\Lambda(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{u},\boldsymbol{v}) + (\mathrm{div}\ \boldsymbol{u}, \mathrm{div}\ \boldsymbol{v}) \tag{2.1}$$

and we let

$$||\ \boldsymbol{v}\ ||_{\mathrm{div}} = \Lambda(\boldsymbol{v},\boldsymbol{v})^{1/2}$$

For domains $K\subset\mathbb{R}^2$, different from Ω, we will in general use an extra subscript to denote norms with respect to the domain K, for example, $||\cdot||_{0,K}$ and $||\cdot||_{\mathrm{div},K}$ will be the norms in $L^2(K)$ and $\boldsymbol{H}(\mathrm{div};K)$, respectively.

The mixed weak formulation of the problem (1.1) is given by:

Find $(\boldsymbol{u},p)\in H(\mathrm{div};\Omega)\times L^2(\Omega)$ such that

$$\begin{aligned} (a^{-1}\,\boldsymbol{u},\boldsymbol{v}) + (p,\mathrm{div}\ \boldsymbol{v}) &= G(\boldsymbol{v}), &&\text{for all } \boldsymbol{v}\in H(\mathrm{div};\Omega)\\ (\mathrm{div}\ \boldsymbol{u}, q) &= F(q), &&\text{for all } q\in L^2(\Omega) \end{aligned} \tag{2.2}$$

Here $\boldsymbol{u}$ is the auxiliary variable $a\,\mathbf{grad}\,p$. The functionals F and G are given by

$$F(q) = -\int_\Omega f q \,\mathrm{d}x \quad \text{and} \quad G(\boldsymbol{v}) = \int_{\partial\Omega} g \;\; \boldsymbol{v}\cdot\boldsymbol{n} \;\mathrm{d}s,$$

where s denotes the arc length along $\partial\Omega$ and $\boldsymbol{n}$ is the exterior unit normal vector on $\partial\Omega$. A key observation for the discussion in this paper is the well known fact that the Dirichlet boundary conditions are natural boundary conditions in the mixed weak formulation.

The system (2.2) can alternatively be written in operator form as

$$\mathscr{A}\begin{pmatrix}\boldsymbol{u}\\ p\end{pmatrix} = \begin{pmatrix}\boldsymbol{G}\\ F\end{pmatrix}$$

where the coefficient operator, $\mathscr{A}$, is given by

$$\mathscr{A} = \begin{pmatrix} a^{-1}\,\boldsymbol{I} & -\mathbf{grad} \\ \mathrm{div} & 0 \end{pmatrix} \tag{2.3}$$

The operator $\mathscr{A}$ can be seen to be an isomorphism mapping $H(\mathrm{div};\,\Omega)\times L^2(\Omega)$ into its L^2–dual $H(\mathrm{div};\,\Omega)^* \times L^2(\Omega)$. A corresponding mapping property for the discrete coefficient operator will be the main tool in deriving the structure of effective preconditioners for the corresponding discrete systems.

The mixed finite element method is derived from the weak formulation (2.2). Let $\{(\mathbf{V}_h, W_h)\}_{h\in(0,1]}$ be pairs of finite element spaces, derived from a family of triangulations $\{\mathcal{T}_h\}_{h\in(0,1]}$, such that $\mathbf{V}_h \subset H(\mathrm{div};\,\Omega)$ and $W_h \subset L^2(\Omega)$ where h denotes the characteristic mesh size. The discrete weak formulation is to find $(\boldsymbol{u}_h, p_h) \in \mathbf{V}_h \times W_h$ such that

$$\begin{aligned} (a^{-1}\,\boldsymbol{u}_h,\,\boldsymbol{v}) + (p_h, \mathrm{div}\;\boldsymbol{v}) &= G(\boldsymbol{v}), &&\text{for all } \;\boldsymbol{v}\in\mathbf{V}_h,\\ (\mathrm{div}\;\boldsymbol{u}_h, q) &= F(q), &&\text{for all } q \in W_h. \end{aligned} \tag{2.4}$$

This system can similarly be written in the operator form

$$\mathscr{A}_h\begin{pmatrix}\boldsymbol{u}_h\\ p_h\end{pmatrix} = \begin{pmatrix}\boldsymbol{G}_h\\ F_h\end{pmatrix}$$

where the operator $\mathscr{A}_h : \mathbf{V}_h \times W_h \mapsto \mathbf{V}_h \times W_h$ has the block structure

$$\mathscr{A}_h = \begin{pmatrix} \boldsymbol{A}_h & \boldsymbol{B}_h^* \\ B_h & 0 \end{pmatrix} \tag{2.5}$$

Here the operators $\boldsymbol{A}_h : \mathbf{V}_h \mapsto \mathbf{V}_h$ and $B_h : \mathbf{V}_h \mapsto W_h$ are defined implicitly by the system (2.4), i.e.,

$$(\boldsymbol{A}_h\boldsymbol{u},\,\boldsymbol{v}) = (a^{-1}\,\boldsymbol{u},\,\boldsymbol{v}) \quad \text{for all } \;\boldsymbol{u},\,\boldsymbol{v}\in\mathbf{V}_h,$$

and

$$(B_h\;\boldsymbol{u}, q) = (\mathrm{div}\;\boldsymbol{u}, q) \quad \text{for all } \;\boldsymbol{u}\in\mathbf{V}_h, q\in W_h.$$

Furthermore, $\boldsymbol{B}_h^*$: $W_h \mapsto \mathbf{V}_h$ is the L^2–adjoint of B_h.

The family of spaces $\{(\mathbf{V}_h, W_h)\}$ are assumed to satisfy a Babuška–Brezzi condition of the form

$$\sup_{\boldsymbol{v}\in\mathbf{V}_h} \frac{(q, \operatorname{div}\, \boldsymbol{v})}{||\, \boldsymbol{v}\, ||_{\mathrm{div}}} \geq \alpha ||q||_0 \quad \text{for all } q \in W_h, \tag{2.6}$$

where $\alpha > 0$ is independent of h. In addition, we assume that

$$\operatorname{div}(\mathbf{V}_h) \subset W_h. \tag{2.7}$$

It is well known that the two conditions (2.6) and (2.7) imply the stability of the mixed finite element method for second order elliptic problems of the form (1.1). More precisely, let $X_h = \mathbf{V}_h \times W_h$ with norm

$$||(\boldsymbol{v}, q)||^2_{X_h} = ||\, \boldsymbol{v}\, ||^2_{\mathrm{div}} + ||q||^2_0,$$

and let X_h^* be equal to X_h as a set, but with the corresponding dual norm, i.e.,

$$||(\boldsymbol{v}, q)||^2_{X_h^*} = ||\, \boldsymbol{v}\, ||^2_* + ||q||^2_0,$$

where

$$||\, \boldsymbol{v}\, ||_* = \sup_{\mathbf{z}\in\mathbf{V}_h} \frac{(\boldsymbol{v}, \boldsymbol{z})}{||\, \boldsymbol{z}\, ||_{\mathrm{div}}}.$$

The conditions (2.6) and (2.7) imply that the operator norms

$$||\mathcal{A}_h||_{\mathcal{L}(X_h, X_h^*)} \quad \text{and} \quad ||\mathcal{A}_h^{-1}||_{\mathcal{L}(X_h^*, X_h)} \quad \text{are bounded uniformly in } h. \tag{2.8}$$

This qualitative description of the system (2.4) is the basic information needed in order to construct the structure of possible preconditioners for the system.

3. Preconditioning mixed systems

Consider the discrete system (2.4) with coefficient operator $\mathcal{A}_h$ given by (2.5). Our aim is to solve this system by an iterative method. However, since $\mathcal{A}_h$ arises as a discretization of the unbounded operator (2.3), a preconditioning of the system is necessary in order to obtain an effective iteration.

The coefficient operator $\mathcal{A}_h$ of the discrete mixed system is L^2–symmetric, but indefinite. For a short review of various iterative methods proposed for saddle point problems we refer to Section 7 of [3] and references given there. In the numerical experiments, presented in Section 6, we will use the minimum residual method. For discussions on the minimum residual method and block diagonal preconditioners for saddle point problems we refer to [19,20,21,31,32,36] and Chapter 9 [16]. The derivation given below of the desired structure of the preconditioner for the saddle point operator is closely related to similar discussions given in [2] and [3], see also [34].

A preconditioner for the system (2.4) is a positive definite operator $\mathcal{B}_h : X_h \mapsto X_h$. The basic properties required for an effective preconditioner $\mathcal{B}_h$ is that the action of $\mathcal{B}_h$ can be efficiently computed and that the spectral condition number of the operator $\mathcal{B}_h\mathcal{A}_h$,

$\kappa(\mathcal{B}_h\mathcal{A}_h)$, is bounded uniformly in h. Here $\kappa(\mathcal{B}_h\mathcal{A}_h)$ is given by

$$\kappa(\mathcal{B}_h\mathcal{A}_h) = \frac{\sup|\lambda|}{\inf|\lambda|}$$

where the supremum and infimum is taken over the spectrum of $\mathcal{B}_h\mathcal{A}_h$. We observe that $\mathcal{B}_h\mathcal{A}_h$ is symmetric with respect to the innerproduct $(\mathcal{B}_h^{-1}\cdot,\cdot)$. Furthermore, this operator is the coefficient operator of the preconditioned system

$$\mathcal{B}_h\mathcal{A}_h\begin{pmatrix}\boldsymbol{u}_h\\ p_h\end{pmatrix} = \mathcal{B}_h\begin{pmatrix}\boldsymbol{G}_h\\ F_h\end{pmatrix}$$

If the condition number $\kappa(\mathcal{B}_h\mathcal{A}_h)$ is bounded, uniformly in h, then an iterative method like the minimum residual method, or another method with similar properties, will converge with a rate independent of h. More precisely, the number of iterations needed to achieve a given factor of reduction of the error in a proper norm is bounded independently of h.

As a consequence of (2.8) the desired bound on the condition number $\kappa(\mathcal{B}_h\mathcal{A}_h)$ will hold if

$$||\mathcal{B}_h||_{\mathcal{L}(X_h^*,X_h)} \quad \text{and} \quad ||\mathcal{B}_h^{-1}||_{\mathcal{L}(X_h,X_h^*)} \quad \text{are bounded uniformly in } h. \tag{3.1}$$

Hence, we conclude that a uniform preconditioner $\mathcal{B}_h$ for the mixed system 2.4) is a positive definite operator $\mathcal{B}_h : X_h \mapsto X_h$ which satisfies the bounds (3.1) and which is easy to evaluate.

Let $\boldsymbol{\Lambda}_h{:}\mathbf{V}_h{\mapsto}\mathbf{V}_h$ be defined by (cf. (2.1))

$$\Lambda(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{\Lambda}_h\boldsymbol{u},\boldsymbol{v}) \quad \text{forall } \boldsymbol{u},\boldsymbol{v}\in\mathbf{V}_h\ .$$

It is straightforward to see that $\boldsymbol{\Lambda}_h$ is L^2–symmetric and positive definite. The operator $\boldsymbol{\Lambda}_h$ should be thought of as an approximation of the operator $\boldsymbol{I} - \mathbf{grad}$ div in the sense that the equation $\boldsymbol{\Lambda}_h\boldsymbol{u}_h = \boldsymbol{f}_h$ approximates problems of the form

$$\boldsymbol{u} - \mathbf{grad}\,\mathrm{div}\,\boldsymbol{u} = \boldsymbol{f}, \tag{3.2}$$

with the natural boundary condition

$$\mathrm{div}\,\boldsymbol{u} = 0 \quad \text{on } \partial\Omega. \tag{3.3}$$

Note that the operator $\boldsymbol{I} - \mathbf{grad}$ div is not an elliptic operator. In fact, when restricted to gradient fields this operator acts like a second order elliptic operator, while it coincides with the identity operator on curl fields.

The significance of the operator $\boldsymbol{\Lambda}_h$ is that the bounds (2.8) state that the operator $\mathcal{A}_h$ has the same mapping properties as the diagonal operator

$$\begin{pmatrix}\boldsymbol{\Lambda}_h & 0\\ 0 & I\end{pmatrix}$$

Furthermore, the simplest choice of an operator which satisfies the condition (3.1) is a block diagonal operator of the form

$$\mathcal{B}_h = \begin{pmatrix}\boldsymbol{\Theta}_h & 0\\ 0 & I\end{pmatrix} \tag{3.4}$$

where $\mathbf{\Theta}_h$ is a uniform preconditioner for $\mathbf{\Lambda}_h$. The operator $\mathbf{\Theta}_h$ should therefore be spectrally equivalent to $\mathbf{\Lambda}_h^{-1}$, i.e., there are positive constants c_1 and c_2 independent of h such that

$$c_1\Lambda(\boldsymbol{v}, \boldsymbol{v}) \le \Lambda(\mathbf{\Theta}_h\mathbf{\Lambda}_h\boldsymbol{v}, \boldsymbol{v}) \le c_2\Lambda(\boldsymbol{v}, \boldsymbol{v}). \tag{3.5}$$

Hence, the construction of a preconditioner $\mathcal{B}_h$ for the coefficient operator $\mathcal{A}_h$ of (2.4) has been reduced to the construction of a preconditioner for the positive definite operator $\mathbf{\Lambda}_h$.

The construction of effective preconditioners for the $\mathbf{\Lambda}_h$ has been discussed by Cai, Goldstein and Pasciak [11], Vassilevski and Wang [35] and Arnold, Falk and Winther [3] in two space dimensions and by Hiptmair [17] and Hiptmair and Tosseli [18] in three dimensions. In particular, it is established in [3] that, if $\mathbf{V}_h$ is a Raviart–Thomas space, a standard multigrid V-cycle operator, utilizing a proper smoothing operator, will in fact be a uniform preconditioner for $\mathbf{\Lambda}_h$. In the analysis below we will discuss the possibility of using a preconditioner constructed with respect to an extended domain to construct a uniform preconditioner for the operator $\mathbf{\Lambda}_h$.

4. Domain embedding

If the geometry of the domain Ω is, in some sense, irregular then for computational reasons it is frequently desirable to embed Ω into a more regular extended domain Ω_e. The purpose of this section is to discuss the use of such an extended domain to construct the preconditioner $\mathbf{\Theta}_h$ for $\mathbf{\Lambda}_h$. We emphasize that even if we are approximating a Dirichlet problem of the form (1.1), the boundary condition given by (3.3) is a natural boundary condition for the problem (3.2). Therefore, the domain embedding approach is well suited for the operator $\mathbf{\Lambda}_h$.

In the first subsection, Section 4.1, we present the setting of domain embedding and prove some auxiliary results, the main one being an extension result of normal trace data into the interior of the neighbouring subdomain Ω_+. Note, that unlike the domain embedding approach studied, e.g., in Nepomnyaschikh [24], we do not need this extension mapping explicitly in the algorithm implementing the actions of the preconditioner. In subsection 4.2 we give the construction of the domain embedding preconditioner and study its spectral equivalence properties.

4.1. An extension result for $\boldsymbol{H}(\mathrm{div})$*–functions*

Let Ω_e be a fixed extended domain, i.e., $\Omega \subset \Omega_e$. The part of $\partial\Omega$ which is contained in the interior of Ω_e will be denoted Γ. For simplicity, we assume that Γ is connected. Furthermore, we let $\Omega_+ = \Omega_e \setminus (\Omega \cup \Gamma)$. Hence,

$$\Omega_e = \Omega \cup \Gamma \cup \Omega_+$$

cf. Figure 1. The inner products in $L^2(\Omega_e)$ and $H(\mathrm{div}, \Omega_e)$ will be denoted $(\cdot,\cdot)_e$ and $\Lambda_e(\cdot,\cdot)$, respectively.

We will use $H_0^1(\Gamma)$ to denote the space

$$H_0^1(\Gamma) = \{\mu|_\Gamma : \mu \in H^1(\partial\Omega), \quad \mu \equiv 0 \text{ on } \partial\Omega \setminus \Gamma\}$$

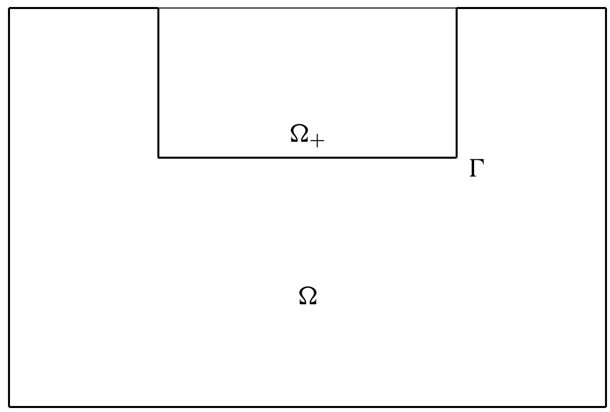

Figure 1. The domain $\Omega_e = \Omega \cup \Gamma \cup \Omega_+$

and $H_0^{1/2}(\Gamma)$ will be the interpolation space half way between $L^2(\Gamma)$ and $H_0^1(\Gamma)$ (This space is frequently referred to as $H_{0,0}^{1/2}(\Gamma)$, cf. [22].) If $K \subset \partial\Omega_e \cup \Gamma$ we let $\langle \cdot, \cdot \rangle_K$ denote the inner product of $L^2(K)$ and $| \cdot |_{m,K}$ the norm in $H^m(K)$. The dual space of $H_0^{\frac{1}{2}}(\Gamma)$ with respect to the inner product $\langle \cdot, \cdot \rangle_\Gamma$ of $L^2(\Gamma)$ is denoted $H^{-1/2}(\Gamma)$.

As above let $\boldsymbol{n} = (n_1, n_2)$ denote the outward unit normal on $\partial\Omega$. It is well known (cf. e.g., [15]) that the trace operator

$$\boldsymbol{v} \longrightarrow (\boldsymbol{v} \cdot \boldsymbol{n})|_\Gamma$$

is a bounded map from $H(\mathrm{div}; \Omega)$ onto $H^{-1/2}(\Gamma)$. More precisely, there exists a positive constant c, only depending on Ω and Γ, such that

$$| \boldsymbol{v} \cdot \boldsymbol{n} |_{-1/2,\Gamma} \leq c || \boldsymbol{v} ||_{\mathrm{div}} \quad \text{forall } \boldsymbol{v} \in H(\mathrm{div}; \Omega) \tag{4.1}$$

We will assume that $\mathbf{V}_h$ can be embedded into a larger finite element space, $\mathbf{V}_h\ (\Omega_e) \subset H(\mathrm{div}; \Omega_e)$. Hence, the space $\mathbf{V}_h$ is obtained by restricting elements of $\mathbf{V}_h\ (\Omega_e)$ to Ω. We recall that if $\boldsymbol{v} \in H(\mathrm{div}; \Omega_e)$ then the normal component of $\boldsymbol{v}$ is required to be continuous over the interface Γ in the sense that

$$(\boldsymbol{v} \cdot \boldsymbol{n})|_{\Gamma+} = (\boldsymbol{v} \cdot \boldsymbol{n})|_{\Gamma-}$$

i.e., the normal components taken from the outside or the inside of Ω are the same. The subspace of functions in $\mathbf{V}_h\ (\Omega_e)$ with support in the closure of Ω_+ will be denoted $\mathbf{V}_{0,h}$, i.e.,

$$\mathbf{V}_{0,h} = \{\boldsymbol{v} \in \mathbf{V}_h\ (\Omega_e) : \boldsymbol{v} \equiv 0 \text{ on } \Omega\}$$

We observe that if $\boldsymbol{v} \in \mathbf{V}_{0,h}$ then $(\boldsymbol{v} \cdot \boldsymbol{n})|_\Gamma = 0$. The orthogonal complement of $\mathbf{V}_{0,h}$ in $\mathbf{V}_h\ (\Omega_e)$ with respect to the inner product $\Lambda_e(\cdot,\cdot)$ will be denoted $\mathbf{V}_{0,h}^\perp$. Hence, $\mathbf{V}_{0,h}^\perp$ consists of all functions in $\mathbf{V}_h\ (\Omega_e)$ which are 'discrete $\mathbf{\Lambda}$–harmonic' on Ω_+.

We shall throughout this section assume that the spaces $\{\mathbf{V}_h\ (\Omega_e)\}$ are Raviart–Thomas spaces constructed from a quasi-uniform triangulation $\{\mathcal{T}_{e,h}\}$ of Ω_e. Hence, for a non-negative integer r,

$$\mathbf{V}_h\ (\Omega_e) = \{\boldsymbol{v} \in H(\mathrm{div}; \Omega_e) : \boldsymbol{v} \in (P_r(T))^2 + (x, y)P_r(T) \quad \text{for all } T \in \mathcal{T}_{e,h}\}$$

Here $P_r(T)$ denotes the space of polynomials of degree at most r on T. The associated spaces W_h and $W_h(\Omega_e)$ consist of discontinuous piecewise polynomials of degree r with respect to the triangulations $\mathcal{T}_h$ and $\mathcal{T}_{e,h}$, respectively. It is well known that the stability condition (2.6) holds for the pairs of spaces $\{(\mathbf{V}_h, W_h)\}$ (cf. [29]).

Let also $Z_h(\Omega_e)$ be the usual space of continuous piecewise polynomials of degree at most $r+1$, i.e.,

$$Z_h(\Omega_e) = \{z \in H^1(\Omega_e) : z|_T \in P_{r+1}(T) \quad \text{for all } T \in \mathcal{T}_{e,h}\}$$

and let **curl** be the differential operator

$$\mathbf{curl} = \begin{pmatrix} -\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} \end{pmatrix}$$

It is easy to check that

$$\mathbf{curl}(Z_h(\Omega_e)) \subset \mathbf{V}_h\ (\Omega_e)$$

In fact, $\mathbf{curl}(Z_h(\Omega_e))$ consists of piecewise polynomials of degree at most r, and $\mathbf{curl}(Z_h(\Omega_e))$ is exactly the subspace of divergence free vector fields in $\mathbf{V}_h\ (\Omega_e)$ (cf. [9]).

Finally, we let $S_h(\Gamma)$ be the trace space given by

$$S_h(\Gamma) = \{\mu : \mu = (\boldsymbol{v} \cdot \boldsymbol{n})|_\Gamma \quad \text{for } \boldsymbol{v} \in \mathbf{V}_h\}$$

Hence, $S_h(\Gamma)$ is a space of discontinuous piecewise polynomials on Γ. Similarly, we let

$$S_h(\partial\Omega_+) = \{\mu : \mu = (\boldsymbol{v} \cdot \boldsymbol{n})|_{\partial\Omega_+} \quad \text{for } \boldsymbol{v} \in \mathbf{V}_h\ (\Omega_e)\}$$

such that $S_h(\Gamma) \subset S_h(\partial\Omega_+)$. In the analysis below we shall utilize the following extension result for boundary functions.

Lemma 4.1. *There exists a bounded extension operator $E_\partial : H^{-1/2}(\Gamma) \mapsto H^{-1/2}(\partial\Omega_+)$ such that*

$$\int_{\partial\Omega_+} (E_\partial g)(s)\, ds = 0 \quad \text{forall} g \in H^{-1/2}(\Gamma)$$

Proof

Let $\gamma_1 > 0$ be the length of Γ and $\gamma_2 > 0$ the length of $\partial\Omega_+ \setminus \Gamma$ such that functions on $\partial\Omega_+$ can be identified with (periodic) functions on $(-\gamma_1, \gamma_2)$. For each $g \in H^{-1/2}(\Gamma) =$

$H^{-1/2}((-\gamma_1, 0))$ define $E_\partial g$ as the odd extension of g given by

$$E_\partial g = \begin{cases} g(s) & \text{for } s \in (-\gamma_1, 0) \\ -\frac{\gamma_1}{\gamma_2} g(-\frac{\gamma_1}{\gamma_2} s) & \text{for } s \in (0, \gamma_2) \end{cases} \tag{4.2}$$

It is straightforward to check that the L^2–dual operator E_∂^* is given by

$$(E_\partial^* \phi)(s) = \phi(s) - \phi(-\frac{\gamma_2}{\gamma_1} s)$$

for $s \in (-\gamma_1, 0)$. Furthermore, the operator E_∂^* is easily seen to satisfy

$$E_\partial^* \in \mathscr{L}(L^2(\partial\Omega_+), L^2(\Gamma)) \cap \mathscr{L}(H^1(\partial\Omega_+), H_0^1(\Gamma))$$

By interpolation we therefore can conclude that

$$E_\partial^* \in \mathscr{L}(H^{1/2}(\partial\Omega_+), H_0^{1/2}(\Gamma))$$

and by duality this implies

$$E_\partial \in \mathscr{L}(H^{-1/2}(\Gamma), H^{-1/2}(\partial\Omega_+))$$

Finally, the mean value property for $E_\partial g$ follows from the fact that $E_\partial^* \phi \equiv 0$ for any constant function ϕ. ■

Let $Q_h : L^2(\partial\Omega_+) \mapsto S_h(\partial\Omega_+)$ be the L^2–projection. Since the space $S_h(\partial\Omega_+)$ in general will contain piecewise constants the operator Q_h admits an estimate of the form

$$|(I - Q_h)\phi|_{0,\partial\Omega_+} \leq ch^\delta |\phi|_{\delta,\partial\Omega_+} \quad \text{for } 0 \leq \delta \leq 1, \tag{4.3}$$

where c is independent of h. We now have the following discrete analogue of Lemma 4.1

Lemma 4.2. *For each $g \in S_h(\Gamma)$ there exists an extension $\tilde{g} \in S_h(\partial\Omega_+)$ such that*

$$\int_{\partial\Omega_+} \tilde{g}(s)\, \mathrm{d}s = 0 \quad \textit{and} \quad |\, \tilde{g}\, |_{-1/2,\partial\Omega_+} \leq c \,|\, g\, |_{-1/2,\Gamma}$$

where the constant c is independent of g and h.

Proof

From Lemma 4.1 it seems that the obvious choice for $\tilde{g}$ is $E_\partial g$. However, in general $E_\partial g \notin S_h(\partial\Omega_+)$ for $g \in S_h(\Gamma)$. But we observe that the definition of E_∂ implies that $E_\partial g$ is a piecewise polynomial with respect to a partition of $\partial\Omega_+$ implicitly defined by (4.2). Therefore, the function $E_\partial g$ will satisfy an inverse estimate of the form

$$|E_\partial g|_{0,\partial\Omega_+} \leq ch^{-1/2} |E_\partial g|_{-1/2,\partial\Omega_+} \leq ch^{-1/2} |g|_{-1/2,\Gamma} \tag{4.4}$$

where the final estimate follows from Lemma 4.1

Define instead $\tilde{g} = Q_h E_\partial g$. Since $S_h(\partial\Omega_+)$ consists of discontinuous functions the operator Q_h is local and therefore

$$g = E_\partial g|_\Gamma = \tilde{g}|_\Gamma$$

Furthermore, since the constant functions are elements of $S_h(\partial\Omega_+)$, the mean value property for $\tilde{g}$ is inherited from the mean value property of $E_\partial g$. Finally, if $\phi \in H^{1/2}(\partial\Omega_+)$ we have

$$\begin{aligned}\langle \tilde{g}, \phi\rangle_{\partial\Omega_+} &= \langle E_\partial g, \phi\rangle_{\partial\Omega_+} + \langle E_\partial g, (Q_h - I)\phi\rangle_{\partial\Omega_+} \\ &\le |E_\partial g|_{-1/2,\partial\Omega_+}|\phi|_{1/2,\partial\Omega_+} + |E_\partial g|_{0,\partial\Omega_+}|(I-Q_h)\phi|_{0,\partial\Omega_+} \\ &\le c|g|_{-1/2,\Gamma}|\phi|_{1/2,\partial\Omega_+},\end{aligned}$$

where we have used the estimates (4.3) and (4.4) in the final step. Hence, it follows from the definition of $|\tilde{g}|_{-1/2,\partial\Omega_+}$ that

$$|\tilde{g}|_{-1/2,\partial\Omega_+} = \sup_{\phi\in H^{1/2}(\partial\Omega_+)} \frac{\langle \tilde{g}, \phi\rangle_{\partial\Omega_+}}{|\phi|_{1/2,\partial\Omega_+}} \le c|g|_{-1/2,\Gamma}.$$

■

Let

$$Z_h(\Omega_+) = \{z|_{\Omega_+} : z \in Z_h(\Omega_e)\}.$$

Lemma 4.3. *(Discrete divergence free extension) Let $g \in S_h(\Gamma)$. There exists a $\boldsymbol{v} \in$* **curl** $(Z_h(\Omega_+))$ *such that*

$$(\boldsymbol{v}\cdot\boldsymbol{n})|_\Gamma = g \qquad \textit{and} \qquad ||\,\boldsymbol{v}\,||_{\mathrm{div},\Omega_+} = ||\,\boldsymbol{v}\,||_{0,\Omega_+} \le c|g|_{-1/2,\Gamma}$$

where c is independent of g and h.

Proof

For a given $g \in S_h(\Gamma)$ let $\tilde{g} \in S_h(\partial\Omega_+)$ be as in Lemma 4.2 Since $\tilde{g}$ has mean value zero, there exists a unique $\phi \in H^{1/2}(\partial\Omega_+)$ such that

$$\phi_t = \tilde{g}, \quad \text{and} \quad \int_{\partial\Omega_+} \phi\, ds = 0$$

Here and in what follows, ϕ_t will denote the tangential derivative of ϕ, where the unit tangent vector $\boldsymbol{t}$ is defined such that $\boldsymbol{t} = (n_2, -n_1)$ on Γ. Furthermore,

$$|\phi|_{1/2,\partial\Omega_+} \le c|\tilde{g}|_{-1/2,\partial\Omega_+} \le c|g|_{-1/2,\Gamma} \tag{4.5}$$

Also, observe that ϕ corresponds to a trace on $\partial\Omega_+$ of a function in $Z_h(\Omega_e)$.

Let ψ be the unique element in $Z_h(\Omega_+)$ such that $\psi = \phi$ on $\partial\Omega_+$ and

$$\int_{\Omega_+} \mathbf{grad}\, \psi \cdot \mathbf{grad}\, z\, \mathrm{d}x = 0$$

for all $z \in Z_h(\Omega_+)$ such that $z|_{\partial\Omega_+} = 0$. Hence, ψ is a ‘discrete harmonic’ function on Ω_+ and therefore (cf. e.g., [7])

$$||\psi||_{1,\Omega_+} \le c|\psi|_{1/2,\partial\Omega_+} = c|\phi|_{1/2,\partial\Omega_+} \tag{4.6}$$

Define now $\boldsymbol{v} = \mathbf{curl}\ \psi$. Then

$$(\boldsymbol{v} \cdot \boldsymbol{n})|_\Gamma = (\psi_t)|_\Gamma = (\phi_t)|_\Gamma = g$$

Furthermore, since

$$||\ \boldsymbol{v}\ ||_{\mathrm{div},\Omega_+} = ||\ \boldsymbol{v}\ ||_{0,\Omega_+} \leq ||\psi||_{1,\Omega_+}$$

the desired estimate follows from (4.5) and (4.6). ■

Lemma 4.4. *There is a constant c, independent of h, such that*

$$||\ \boldsymbol{u}\ ||_{\mathrm{div},\Omega_+} \leq c|\ \boldsymbol{u} \cdot \boldsymbol{n}\ |_{-1/2,\Gamma} \quad \textit{for all}\ \ \boldsymbol{u} \in \mathbf{V}_{0,h}^{\perp}$$

Proof

Let $\boldsymbol{u} \in \mathbf{V}_{0,h}^{\perp}$ and let $g = (\boldsymbol{u} \cdot \boldsymbol{n})|_\Gamma \in S_h(\Gamma)$. Since $\boldsymbol{u} \in \mathbf{V}_{0,h}^{\perp}$ it follows that this function minimizes $||\ \boldsymbol{v}\ ||_{\mathrm{div},\Omega_+}$ over all elements in $\mathbf{V}_h\ (\Omega_e)$ which satisfy the condition $(\boldsymbol{v} \cdot \boldsymbol{n})$ $|_\Gamma = g$. Therefore, it is enough to establish the existence of a $\boldsymbol{v} \in \mathbf{V}_h\ (\Omega_+) = \mathbf{V}_h\ (\Omega_e)|_{\Omega_+}$ such that $(\boldsymbol{v} \cdot \boldsymbol{n})|_\Gamma = g$ and

$$||\ \boldsymbol{v}\ ||_{\mathrm{div},\Omega_+} \leq c|g|_{-1/2,\Gamma} \tag{4.7}$$

However, the existence of such a function $\boldsymbol{v}$ follows from Lemma 4.3 ■

We observe that, together with the trace inequality (4.1), this lemma implies an inequality of the form

$$||\ \boldsymbol{v}\ ||_{\mathrm{div},\Omega_+} \leq c||\ \boldsymbol{v}\ ||_{\mathrm{div}} \quad \text{forall}\ \ \boldsymbol{v} \in \mathbf{V}_{0,h}^{\perp}$$

Hence, since $||\ \boldsymbol{v}\ ||^2_{\mathrm{div},\Omega_e} = ||\ \boldsymbol{v}\ ||^2_{\mathrm{div}} = ||\ \boldsymbol{v}\ ||^2_{\mathrm{div},\Omega_+}$ we obtain

$$||\ \boldsymbol{v}\ ||_{\mathrm{div},\Omega_e} \leq \beta||\ \boldsymbol{v}\ ||_{\mathrm{div}} \quad \text{forall}\ \ \boldsymbol{v} \in \mathbf{V}_{0,h}^{\perp} \tag{4.8}$$

where the positive constant β is independent of h. Also, for each $\boldsymbol{v} \in \mathbf{V}_h$ there exists a unique extension $\tilde{\boldsymbol{v}} \in \mathbf{V}_{0,h}^{\perp}$ such that $\tilde{\boldsymbol{v}} = \boldsymbol{v}$ on Ω. In fact, $\tilde{\boldsymbol{v}}|_{\Omega_+}$ is defined by

$$\begin{aligned} \Lambda_e(\tilde{\boldsymbol{v}}, \boldsymbol{z})\ &= 0 \quad \text{for all}\ \ \boldsymbol{z} \in \mathbf{V}_{0,h} \\ (\tilde{\boldsymbol{v}} \cdot \boldsymbol{n})|_{\Gamma+}\ &= (\boldsymbol{v} \cdot \boldsymbol{n})|_{\Gamma-} \end{aligned} \tag{4.9}$$

Hence, if $\tilde{\boldsymbol{v}} \in \mathbf{V}_{0,h}^{\perp}$ is defined from $\boldsymbol{v} \in \mathbf{V}_h$ by (4.9) then (4.8) implies that

$$||\tilde{\boldsymbol{v}}||_{\mathrm{div},\Omega_e} \leq \beta||\ \boldsymbol{v}\ ||_{\mathrm{div}} \tag{4.10}$$

4.2. The domain embedding preconditioner

In this subsection we present the construction of the domain embedding preconditioner for the operator of main interest, Λ_h, and analyse its properties.

Let $\boldsymbol{R}_h : \mathbf{V}_h\ (\Omega_e) \mapsto \mathbf{V}_h$ be the operator defined by restricting elements of $\mathbf{V}_h\ (\Omega_e)$ to Ω and let $\boldsymbol{E}_h : \mathbf{V}_h \mapsto \mathbf{V}_h\ (\Omega_e)$ be the adjoint operator, i.e.,

$$(\boldsymbol{E}_h \boldsymbol{u},\ \boldsymbol{v})_e = (\boldsymbol{u}, \boldsymbol{R}_h \boldsymbol{v}) \quad \text{for all}\ \ \boldsymbol{u} \in \mathbf{V}_h,\ \boldsymbol{v} \in \mathbf{V}_h\ (\Omega_e)$$

Alternatively, $\boldsymbol{E}_h\boldsymbol{u}$ can be characterized as the L^2–projection into $\mathbf{V}_h\ (\Omega_e)$ of $\boldsymbol{u}^0$, where $\boldsymbol{u}^0 \in L^2(\Omega_e)$ denotes the extension of $\boldsymbol{u}$ by zero outside Ω_e.

The analogue of the operator $\boldsymbol{\Lambda}_h$ with respect to the extended space $\mathbf{V}_h\ (\Omega_e)$ will be denoted $\boldsymbol{\Lambda}_{e,h}$, i.e., $\boldsymbol{\Lambda}_{e,h}:\mathbf{V}_h\ (\Omega_e) \mapsto \mathbf{V}_h\ (\Omega_e)$ is defined by

$$(\boldsymbol{\Lambda}_{e,h}\boldsymbol{u},\ \boldsymbol{v})_e = \Lambda_e(\boldsymbol{u},\ \boldsymbol{v}) \quad \text{for all}\ \ \boldsymbol{u},\ \boldsymbol{v} \in \mathbf{V}_h\ (\Omega_e)$$

We now define $\boldsymbol{T}_h:\mathbf{V}_h \mapsto \mathbf{V}_h\ (\Omega_e)$ by

$$\boldsymbol{T}_h = \boldsymbol{\Lambda}_{e,h}^{-1}\boldsymbol{E}_h\boldsymbol{\Lambda}_h$$

The operator $\boldsymbol{T}_h$ preserves the $H(\mathrm{div})$–inner product in the sense that

$$\Lambda_e(\boldsymbol{T}_h\boldsymbol{u},\ \boldsymbol{v}) = \Lambda(\boldsymbol{u}, \boldsymbol{R}_h\,\boldsymbol{v}) \quad \text{for all}\ \ \boldsymbol{u} \in \mathbf{V}_h,\ \boldsymbol{v} \in \mathbf{V}_h\ (\Omega_e) \tag{4.11}$$

This follows since

$$\begin{aligned} \Lambda_e(\boldsymbol{T}_h\boldsymbol{u},\ \boldsymbol{v}) &= (\boldsymbol{\Lambda}_{e,h}\boldsymbol{T}_h\boldsymbol{u},\ \boldsymbol{v})_e = (\boldsymbol{E}_h\boldsymbol{\Lambda}_h\boldsymbol{u},\ \boldsymbol{v})_e \\ &= (\boldsymbol{\Lambda}_h\boldsymbol{u}, \boldsymbol{R}_h\,\boldsymbol{v}) = \boldsymbol{\Lambda}\ (\boldsymbol{u}, \boldsymbol{R}_h\,\boldsymbol{v}). \end{aligned}$$

Furthermore, if $\boldsymbol{u} \in \mathbf{V}_h$ it follows from (4.11) and the Cauchy–Schwarz inequality that

$$\begin{aligned} \Lambda_e(\boldsymbol{T}_h\boldsymbol{u}, \boldsymbol{T}_h\boldsymbol{u}) &= \Lambda(\boldsymbol{u}, \boldsymbol{R}_h\boldsymbol{T}_h\boldsymbol{u}) \le \Lambda(\boldsymbol{u}, \boldsymbol{u})^{1/2}\Lambda(\boldsymbol{R}_h\boldsymbol{T}_h\boldsymbol{u}, \boldsymbol{R}_h\boldsymbol{T}_h\boldsymbol{u})^{1/2} \\ &\le \Lambda(\boldsymbol{u}, \boldsymbol{u})^{1/2}\Lambda_e(\boldsymbol{T}_h\boldsymbol{u}, \boldsymbol{T}_h\boldsymbol{u})^{1/2} \end{aligned}$$

or

$$\Lambda_e(\boldsymbol{T}_h\boldsymbol{u}, \boldsymbol{T}_h\boldsymbol{u}) \le \Lambda(\boldsymbol{u}, \boldsymbol{u}) \quad \text{for all}\ \ \boldsymbol{u} \in \mathbf{V}_h \tag{4.12}$$

The following result shows that $\boldsymbol{\Lambda}_{e,h}^{-1}$ can be used to construct a uniform preconditioner for $\boldsymbol{\Lambda}_h$.

Lemma 4.5. *The operator $\boldsymbol{R}_h\boldsymbol{\Lambda}_{e,h}^{-1}\boldsymbol{E}_h$ is spectrally equivalent to $\boldsymbol{\Lambda}_h^{-1}$ in the sense that*

$$\beta^{-2}\Lambda(\boldsymbol{v},\ \boldsymbol{v}) \le \Lambda((\boldsymbol{R}_h\boldsymbol{\Lambda}_{e,h}^{-1}\boldsymbol{E}_h)\ \boldsymbol{\Lambda}_h\,\boldsymbol{v},\ \boldsymbol{v}) \le \Lambda(\boldsymbol{v},\ \boldsymbol{v}) \quad \textit{for all}\ \ \boldsymbol{v} \in V_h.$$

The constant β is from estimate (4.10).

Proof

This result is just a special case of ‘the fictitious space lemma’ given in Nepomnyaschikh [26]. However, for completeness we outline the proof in the present setting. The right inequality above follows since

$$\Lambda((\boldsymbol{R}_h\boldsymbol{\Lambda}_{e,h}^{-1}\boldsymbol{E}_h)\ \boldsymbol{\Lambda}_h\,\boldsymbol{v},\ \boldsymbol{v}) = \Lambda(\boldsymbol{R}_h\boldsymbol{T}_h\,\boldsymbol{v},\ \boldsymbol{v}) = \Lambda_e(\boldsymbol{T}_h\,\boldsymbol{v}, \boldsymbol{T}_h\,\boldsymbol{v}) \le \Lambda(\boldsymbol{v},\ \boldsymbol{v})$$

where we have used (4.11) and (4.12).

The left inequality follows from the identity

$$\Lambda(\boldsymbol{v},\ \boldsymbol{v}) = \Lambda(\boldsymbol{v}, \boldsymbol{R}_h\ \tilde{\boldsymbol{v}}) = \Lambda_e(\boldsymbol{T}_h\,\boldsymbol{v},\ \tilde{\boldsymbol{v}}),$$

where $\tilde{\boldsymbol{v}} \in \mathbf{V}_{0,h}^{\perp}$ is the extension of $\boldsymbol{v}$ defined by (4.9). Therefore, by (4.10), (4.11) and the

Cauchy–Schwarz inequality

$$\begin{aligned}\Lambda(\boldsymbol{v},\boldsymbol{v}) &\le \Lambda_e(\boldsymbol{T}_h\boldsymbol{v},\boldsymbol{T}_h\boldsymbol{v})^{1/2}\Lambda_e(\tilde{\boldsymbol{v}},\tilde{\boldsymbol{v}})^{1/2}\\ &\le \beta\Lambda_e(\boldsymbol{T}_h\boldsymbol{v},\boldsymbol{T}_h\boldsymbol{v})^{1/2}\Lambda(\boldsymbol{v},\boldsymbol{v})^{1/2}\\ &= \beta\Lambda((\boldsymbol{R}_h\boldsymbol{\Lambda}_{e,h}^{-1}\boldsymbol{E}_h)\,\boldsymbol{\Lambda}_h\boldsymbol{v},\boldsymbol{v})^{1/2}\Lambda(\boldsymbol{v},\boldsymbol{v})^{1/2}.\end{aligned}$$

This completes the proof. ■

The lemma above shows that $\boldsymbol{\Lambda}_{e,h}^{-1}$ can be used to construct a uniform preconditioner for $\boldsymbol{\Lambda}_h$. However, it is clear that it suffices to use a preconditioner $\boldsymbol{\Theta}_{e,h}$ instead of $\boldsymbol{\Lambda}_{e,h}$.

Assume that $\boldsymbol{\Theta}_{e,h}:\mathbf{V}_h\ (\Omega_e)\mapsto\mathbf{V}_h\ (\Omega_e)$ is L^2–symmetric and spectrally equivalent to $\boldsymbol{\Lambda}_{e,h}^{-1}$, i.e.

$$c_1(\boldsymbol{\Lambda}_{e,h}^{-1}\boldsymbol{v},\boldsymbol{v}) \le (\boldsymbol{\Theta}_{e,h}\boldsymbol{v},\boldsymbol{v}) \le c_2(\boldsymbol{\Lambda}_{e,h}^{-1}\boldsymbol{v},\boldsymbol{v}) \quad \text{for all} \quad \boldsymbol{v}\in\mathbf{V}_h\ (\Omega_e) \tag{4.13}$$

where the positive constants c_1 and c_2 are independent of h. Define an operator $\boldsymbol{\Theta}_h:\mathbf{V}_h\mapsto\mathbf{V}_h$ by

$$\boldsymbol{\Theta}_h = \boldsymbol{R}_h\boldsymbol{\Theta}_{e,h}\boldsymbol{E}_h \tag{4.14}$$

Observe that $\boldsymbol{\Theta}_h$ is L^2–symmetric since

$$(\boldsymbol{\Theta}_h\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{\Theta}_{e,h}\boldsymbol{E}_h\boldsymbol{u},\boldsymbol{E}_h\boldsymbol{v})_e = (\boldsymbol{u},\boldsymbol{\Theta}_h\boldsymbol{v}) \quad \text{for all} \quad \boldsymbol{u},\boldsymbol{v}\in\mathbf{V}_h$$

The following theorem is the main result of this section.

Theorem 4.1. *Assume that the L^2–symmetric operator $\boldsymbol{\Theta}_{e,h}:\mathbf{V}_h\ (\Omega_e)\mapsto\mathbf{V}_h\ (\Omega_e)$ satisfies (4.13) and let $\boldsymbol{\Theta}_h$: $V_h\mapsto V_h$ be defined by (4.14). Then the two quadratic forms*

$$\Lambda(\boldsymbol{v},\boldsymbol{v}) \quad \textit{and} \quad \Lambda(\boldsymbol{\Theta}_h\boldsymbol{\Lambda}_h\boldsymbol{v},\boldsymbol{v})$$

are equivalent on $\mathbf{V}_h$, uniformly in h.

Proof
We observe that for any $\boldsymbol{v}\in\mathbf{V}_h$

$$\Lambda(\boldsymbol{\Theta}_h\boldsymbol{\Lambda}_h\boldsymbol{v},\boldsymbol{v}) = (\boldsymbol{\Theta}_{e,h}\boldsymbol{E}_h\boldsymbol{\Lambda}_h\boldsymbol{v},\boldsymbol{E}_h\boldsymbol{\Lambda}_h\boldsymbol{v})$$

Furthermore, by (4.13) this quadratic form is equivalent, uniformly in h, to the quadratic form

$$(\boldsymbol{\Lambda}_{e,h}^{-1}\boldsymbol{E}_h\boldsymbol{\Lambda}_h\boldsymbol{v},\boldsymbol{E}_h\boldsymbol{\Lambda}_h\boldsymbol{v}) = \Lambda((\boldsymbol{R}_h\boldsymbol{\Lambda}_{e,h}^{-1}\boldsymbol{E}_h)\,\boldsymbol{\Lambda}_h\boldsymbol{v},\boldsymbol{v}),$$

and by Lemma 4.5 this form is equivalent to $\Lambda(\boldsymbol{v},\boldsymbol{v})$. ■

Hence we have demonstrated that a preconditioner $\boldsymbol{\Theta}_h$, constructed from the extended domain Ω_e as indicated by (4.13) and (4.14), is spectrally equivalent to $\boldsymbol{\Lambda}_h^{-1}$. Therefore, it follows from the discussion given in Section 3 above that a preconditioner $\mathcal{B}_h$ for the mixed system (2.4), derived from the Dirichlet problem (1.1), is naturally constructed by the domain embedding approach.

5. The non-conforming Crouzeix–Raviart method

The purpose of this section is to adopt the preconditioning strategy discussed above for the mixed finite element method to an alternative discretization procedure which is not directly based on the mixed weak formulation (2.2). More precisely, we shall analyse the non-conforming Crouzeix–Raviart method (cf. [12] or Chapter 8 [8]). We will show that this system can be interpreted as a nonconforming mixed system. Furthermore, the obtained saddle point problem will be preconditioned by utilizing a preconditioner of the form studied in Section 4 above, on the lowest order Raviart–Thomas space, combined with the the *auxiliary space method* (cf. Xu [37]). Hence, since we have established above that the preconditioners studied in Section 4 can be naturally constructed by domain embedding, this shows, implicitly, that also the Crouzeix–Raviart system can be preconditioned by domain embedding.

Hence, the idea is to reformulate the positive definite Crouzeix–Raviart system as a saddle point problem, precondition the saddle point problem and then solve the preconditioned system by an iterative method for saddle point problems. This is in contrast to the more common approach where saddle point problems are reduced to positive definite problems in order to apply standard iterative techniques like the conjugate gradient method.

5.1. The non-conforming method as a mixed system

As above, let $\{\mathcal{T}_h\}_{h\in(0,1]}$ be a quasi-uniform family of triangulations of Ω. Furthermore, $\mathcal{E}_h$ is the set of edges in $\mathcal{T}_h$ and for each $E \in \mathcal{E}_h$, x_E is the midpoint of the edge. Throughout this section we let

$$W_h^0 = \{w : w \in P_1(T)\ \forall T \in \mathcal{T}_h,\ w \text{ is continuous at } x_E\ \forall E \in \mathcal{E}_h\}$$

Here the continuity requirement at the boundary should be interpreted such that $w(x_E) = 0$ for all boundary edges E.

We consider approximations of the Dirichlet problem (1.1) with homogeneous boundary conditions. The discrete solutions are determined by:

Find $p_h \in W_h^0$ such that

$$\sum_{T\in\mathcal{T}_h} \int_T (a\ \mathbf{grad}\ p_h)\cdot(\mathbf{grad}\ q_h)\,\mathrm{d}x = (f, q) \quad \text{for all } q \in W_h^0 \tag{5.1}$$

For simplicity we will assume throughout this section that the coefficient matrix a is piecewise constant with respect to the triangulation $\mathcal{T}_h$ for all $h \in (0, 1]$. (Otherwise, just replace a by its average on each triangle in the discussion below.) Let $\mathbf{V}_h^0$ denote the space of discontinuous piecewise constant 2–vectorfields with respect to the triangulation $\mathcal{T}_h$. Define an operator $\mathbf{grad}_h : W_h^0 \mapsto \mathbf{V}_h^0$ by taking the gradient elementwise, i.e., $(\mathbf{grad}_h q)|_T = \mathbf{grad}(q|_T)$. Then the equation (5.1) can be equivalently written as

$$(a\mathbf{grad}_h p_h, \mathbf{grad}_h q) = (f, q) \quad \text{for all } q \in W_h^0$$

We also define a discrete divergence operator $\mathrm{div}_h : \mathbf{V}_h^0 \mapsto W_h^0$ by duality, i.e.,

$$(\mathrm{div}_h\ \boldsymbol{v}, q) = -(\boldsymbol{v}, \mathbf{grad}_h q) \quad \text{for all } q \in W_h^0$$

A more explicit characterization of the operator div_h can be derived from the fact that $\mathrm{div}(\boldsymbol{v}\,|_T) = 0$ for any $\boldsymbol{v} \in \mathbf{V}_h^0$. For each $E \in \mathscr{E}_h$ let $\boldsymbol{n}$ be chosen unit normal vector. If $\boldsymbol{v} \in \mathbf{V}_h^0$ then $[\boldsymbol{v} \cdot \boldsymbol{n}]_E$ will denote the jump of $\boldsymbol{v}$ in this direction. Then we have

$$(\mathrm{div}_h\ \boldsymbol{v}, q) = \sum_{E \in \mathscr{E}_{0,h}} |E|\, q(x_E)\, [\boldsymbol{v} \cdot \boldsymbol{n}]_E \quad \text{for all } q \in W_h^0 \tag{5.2}$$

where $\mathscr{E}_{0,h}$ denotes the set of interior edges and $|E|$ is the length of E.

If $p_h \in W_h^0$ is the solution of 5.1 we let the discrete flux $\boldsymbol{u}_h \in \mathbf{V}_h^0$ be given by

$$\int_\Omega a^{-1}\, \boldsymbol{u}_h \cdot \boldsymbol{v}\ \mathrm{d}x = \int_\Omega \mathbf{grad}_h\, p_h \cdot \boldsymbol{v}\ \mathrm{d}x \quad \text{for all } \boldsymbol{v} \in \mathbf{V}_h^0$$

The two unknowns $\boldsymbol{u}_h$ and p_h can now be determined as the unique solution $(\boldsymbol{u}_h, p_h) \in \mathbf{V}_h^0 \times W_h^0$ of the saddle point problem

$$\begin{aligned} (a^{-1}\, \boldsymbol{u}_h, \boldsymbol{v}) - (\mathbf{grad}_h\, p_h, \boldsymbol{v}) &= 0, \quad \text{for all } \boldsymbol{v} \in \mathbf{V}_h^0, \\ (\mathrm{div}_h\ \boldsymbol{u}_h, q) &= -(f, q), \quad \text{for all } q \in W_h^0. \end{aligned} \tag{5.3}$$

We note that, under the assumption that the coefficient matrix a is piecewise constant, the system (5.3) is exactly equivalent to the original system (5.1).

We observe that the system (5.3) can be given the operator form:

$$\mathscr{A}_h \begin{pmatrix} \boldsymbol{u}_h \\ p_h \end{pmatrix} = \begin{pmatrix} 0 \\ F_h \end{pmatrix}$$

where the coefficient operator, $\mathscr{A}_h : X_h \mapsto X_h$, is given by

$$\mathscr{A}_h = \begin{pmatrix} a^{-1}\, \boldsymbol{I} & -\mathbf{grad}_h \\ \mathrm{div}_h & 0 \end{pmatrix}$$

Here $X_h = \mathbf{V}_h^0 \times W_h^0$.

If $\boldsymbol{v} \in \mathbf{V}_h^0$ we define a mesh dependent '$H(\mathrm{div})$–norm' by

$$\|\ \boldsymbol{v}\ \|_{\mathrm{div},h}^2 = \|\ \boldsymbol{v}\ \|_0^2 + \|\mathrm{div}_h\ \boldsymbol{v}\ \|_0^2$$

Furthermore, in analogy with the notation above for the standard mixed method, we let

$$\Lambda_h(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{v}) + (\mathrm{div}_h\ \boldsymbol{u}, \mathrm{div}_h\ \boldsymbol{v})$$

be the corresponding inner product.

Lemma 5.1. *For each $q \in W_h^0$ there is a $\boldsymbol{v} \in \mathbf{V}_h^0$ such that*

$$\mathrm{div}_h\ \boldsymbol{v} = q \qquad \textit{and} \qquad \|\ \boldsymbol{v}\ \|_{\mathrm{div},h} \le c\|q\|_0$$

where the constant c is independent of q and h.

Proof
We first recall that the functions in $w \in W_h^0$ satisfy a discrete Poincaré inequality of the form

$$||w||_0 \leq c||\mathbf{grad}_h w||_0 \tag{5.4}$$

where c is independent of w and h. A proof of this fact can for example be found in [1] (cf. Lemma 5.3 of [1]). For a given $q \in W_h^0$ let $\phi \in W_h^0$ be uniquely determined by

$$(\mathbf{grad}_h \phi, \mathbf{grad}_h w) = -(q, w) \quad \text{for all } w \in W_h^0 \tag{5.5}$$

and let $\boldsymbol{v} = \mathbf{grad}_h \phi \in \mathbf{V}_h^0$. By construction $\mathrm{div}_h \ \boldsymbol{v} = q$. Furthermore, 5.4 and (5.5) imply that

$$|| \ \boldsymbol{v} \ ||_0^2 \leq ||q||_0 ||\phi||_0 \leq c||q||_0|| \ \boldsymbol{v} \ ||_0$$

and hence the desired bound on $|| \ \boldsymbol{v} \ ||_{\mathrm{div},h}$ is established. ■

It is a direct consequence of Lemma 5.1 that a Babuška–Brezzi condition of the form

$$\inf_{q \in W_h^0} \sup_{\boldsymbol{v} \in \mathbf{V}_h^0} \frac{(\mathrm{div}_h \ \boldsymbol{v}, q)}{|| \ \boldsymbol{v} \ ||_{\mathrm{div},h} ||q||_0} \geq \alpha > 0$$

holds, where α is independent of h. Therefore, if we let

$$||(\boldsymbol{v}, q)||_{X_h}^2 = || \ \boldsymbol{v} \ ||_{\mathrm{div},h}^2 + ||q||_0^2$$

and, if the dual norm $||(\boldsymbol{v}, q))||_{X_h^*}$ on X_h is defined by L^2–duality, we immediately obtain that the operator norms

$$||\mathscr{A}_h||_{\mathscr{L}(X_h, X_h^*)} \quad \text{and} \quad ||\mathscr{A}_h^{-1}||_{\mathscr{L}(X_h^*, X_h)} \quad \text{are bounded uniformly in } h \tag{5.6}$$

Hence, the properties of the operator $\mathscr{A}_h$ correspond to similar properties for the coefficient operator of the standard mixed method studied above (cf. 2.8). By arguing exactly as we did above we therefore conclude that if $\mathscr{B}_h : X_h \mapsto X_h$ is a positive definite operator such that

$$||\mathscr{B}_h||_{\mathscr{L}(X_h^*, X_h)} \quad \text{and} \quad ||\mathscr{B}_h^{-1}||_{\mathscr{L}(X_h, X_h^*)} \quad \text{are bounded uniformly in } h \tag{5.7}$$

then the condition number of the operator $\mathscr{B}_h \mathscr{A}_h$ is bounded uniformly in h.

Let $\boldsymbol{\Lambda}_h^0 : \mathbf{V}_h^0 \mapsto \mathbf{V}_h^0$ be defined by

$$(\boldsymbol{\Lambda}_h^0 \boldsymbol{u}, \ \boldsymbol{v}) = \Lambda_h(\boldsymbol{u}, \ \boldsymbol{v}) \quad \text{for all } \ \boldsymbol{u}, \ \ \boldsymbol{v} \in \mathbf{V}_h^0$$

This operator is L^2–symmetric and positive definite. Furthermore, assume that $\boldsymbol{\Theta}_h^0 : \mathbf{V}_h^0 \mapsto \mathbf{V}_h^0$ is a uniform preconditioner for $\boldsymbol{\Lambda}_h^0$, and that $I_h : W_h^0 \mapsto W_h^0$ is spectrally equivalent to the identity. If $\mathscr{B}_h : X_h \mapsto X_h$ is the block diagonal operator

$$\mathscr{B}_h = \begin{pmatrix} \boldsymbol{\Theta}_h^0 & 0 \\ 0 & I_h \end{pmatrix}$$

then this operator satisfies the mapping property (5.7). Hence, the construction of a preconditioner $\mathscr{B}_h$ is essentially reduced to the problem of constructing an effective preconditioner

$\mathbf{\Theta}_h^0$ for $\mathbf{\Lambda}_h^0$. Such a preconditioner will be constructed below by the auxiliary space method. The operator I_h is introduced as a replacement for the identity operator, in order to avoid the inversion of a 'mass–matrix'.

5.2. The auxiliary space technique

Let the spaces $\mathbf{V}_h^0$ and W_h^0 be as above, i.e., $\mathbf{V}_h^0$ is the space of discontinuous piecewise constant vectors and W_h^0 is the non-conforming Crouzeix–Raviart space. Furthermore, $\mathbf{V}_h \subset H(\text{div}; \Omega)$ will denote the corresponding lowest order Raviart–Thomas space as described in Section 4 above, i.e. the parameter $r = 0$, and W_h the corresponding space for the mixed method consists of piecewise constants.

In the auxiliary space method the main tool for constructing a preconditioner $\mathbf{\Theta}_h^0 : \mathbf{V}_h^0 \mapsto \mathbf{V}_h^0$ for the operator $\mathbf{\Lambda}_h^0$ is a corresponding preconditioner $\mathbf{\Theta}_h$ for the operator $\mathbf{\Lambda}_h$ defined on the auxiliary space $\mathbf{V}_h$. Here we recall that $\mathbf{\Lambda}_h : \mathbf{V}_h \mapsto \mathbf{V}_h$ is defined by

$$\Lambda(\boldsymbol{u}, \boldsymbol{v}) = (\mathbf{\Lambda}_h \boldsymbol{u}, \boldsymbol{v}) \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in \mathbf{V}_h$$

where $\Lambda(\cdot, \cdot)$ is the $H(\text{div})$–inner product. The preconditioner $\mathbf{\Theta}_h : \mathbf{V}_h \mapsto \mathbf{V}_h$ is assumed to be a uniform preconditioner for $\mathbf{\Lambda}_h$, i.e., the bilinear forms

$$\Lambda(\boldsymbol{v}, \boldsymbol{v}) \qquad \text{and} \qquad \Lambda(\mathbf{\Theta}_h \mathbf{\Lambda}_h \boldsymbol{v}, \boldsymbol{v})$$

are equivalent on $\mathbf{V}_h$, uniformly in h. Since the operators $\mathbf{\Lambda}_h$ and $\mathbf{\Lambda}_h^0$ are finite element approximations of the same differential operator $\mathbf{\Lambda}$, preconditioners for these operators must be related. This observation is utilized in the construction of the auxiliary space preconditioners. Furthermore, in the present setting it is crucial that the subspaces of divergence free vector fields in $\mathbf{V}_h$ and $\mathbf{V}_h^0$ coincide.

In the rest of this section we assume that $\mathbf{\Theta}_h \boldsymbol{v}$ can be effectively evaluated from a given inner product representation of $\boldsymbol{v}$, i.e. from the data $(\boldsymbol{v}, \boldsymbol{\phi}_j)$, where $\{\boldsymbol{\phi}_j\}$ is a nodal basis for $\mathbf{V}_h$. Let $\mathbf{\Pi}_h : \mathbf{V}_h \mapsto \mathbf{V}_h^0$ be the L^2–projection. Note that this operator is local, since $\mathbf{V}_h^0$ is a space of discontinous functions. Furthermore, let $\mathbf{\Pi}_h^* : \mathbf{V}_h^0 \mapsto \mathbf{V}_h$ be the L^2–dual operator, i.e., $\mathbf{\Pi}_h^*$ is the L^2–projection onto $\mathbf{V}_h$.

The auxiliary space preconditioner $\mathbf{\Theta}_h^0 : \mathbf{V}_h^0 \mapsto \mathbf{V}_h^0$ is of the form

$$\mathbf{\Theta}_h^0 = \tau h^2 \boldsymbol{I} + \mathbf{\Pi}_h \mathbf{\Theta}_h \mathbf{\Pi}_h^* \tag{5.8}$$

where τ is a positive constant independent of h. Let us first remark that this operator is computationally feasible. This just follows from the assumption on $\mathbf{\Theta}_h$ together with the fact that $\mathbf{\Pi}_h$ is local. In order to use the theory developed in [37] we need to verify that the projection $\mathbf{\Pi}_h$ is stable and accurate in the sense that

$$\Lambda_h(\mathbf{\Pi}_h \boldsymbol{v}, \mathbf{\Pi}_h \boldsymbol{v}) \leq c \Lambda(\boldsymbol{v}, \boldsymbol{v}) \qquad \text{forall } \boldsymbol{v} \in \mathbf{V}_h \tag{5.9}$$

and

$$||(\boldsymbol{I} - \mathbf{\Pi}_h)\ \boldsymbol{v}\ ||_0 \leq ch||\text{div}\ \boldsymbol{v}\ ||_0 \qquad \text{forall } \boldsymbol{v} \in \mathbf{V}_h \tag{5.10}$$

for a suitable constant c independent of h. Furthermore, we need to construct an operator

$\boldsymbol{P}_h{:}\mathbf{V}_h^0 \mapsto \mathbf{V}_h$ such that

$$\Lambda(\boldsymbol{P}_h\boldsymbol{w}, \boldsymbol{P}_h\boldsymbol{w}) \le c\Lambda_h(\boldsymbol{w}, \boldsymbol{w}) \qquad \text{for all } \boldsymbol{w} \in \mathbf{V}_h^0 \tag{5.11}$$

and

$$||(\boldsymbol{I} - \boldsymbol{P}_h)\,\boldsymbol{w}\,||_0 \le ch||\mathrm{div}_h\,\boldsymbol{w}\,||_0 \qquad \text{for all } \boldsymbol{w} \in \mathbf{V}_h^0 \tag{5.12}$$

where again c is independent of h. The operator $\boldsymbol{P}_h$ is only needed for the analysis.

In the present setting we define $\boldsymbol{P}_h{:}\mathbf{V}_h^0 \mapsto \mathbf{V}_h$ by averaging the normal components on each edge, i.e.

$$(\boldsymbol{P}_h\boldsymbol{w}\cdot\boldsymbol{n})_E = \frac{1}{2}\left((\boldsymbol{w}\cdot\boldsymbol{n})_{E-} + (\boldsymbol{w}\cdot\boldsymbol{n})_{E+}\right) \qquad \text{for all } E \in \mathscr{E}_h$$

This will uniquely determine $\boldsymbol{P}_h\boldsymbol{w} \in \mathbf{V}_h$.

Lemma 5.2. *The operators $\mathbf{\Pi}_h$ and $\boldsymbol{P}_h$ defined above satisfy the properties (5.9)–(5.12).*

We will delay the verification of these properties. However, the following theorem now follows more or less directly from [37].

Theorem 5.1. *Let $\mathbf{\Theta}_h^0{:}\mathbf{V}_h^0 \mapsto \mathbf{V}_h^0$ be defined by (5.8). If $\mathbf{\Theta}_h$ is a uniform preconditioner for $\mathbf{\Lambda}_h$ then $\mathbf{\Theta}_h^0$ is a uniform preconditioner for $\mathbf{\Lambda}_h^0$.*

Proof
For completeness we outline a proof in the present setting. We need to show that the bilinear forms

$$\Lambda_h(\boldsymbol{w}, \boldsymbol{w}) \qquad \text{and} \qquad \Lambda_h(\mathbf{\Theta}_h^0\mathbf{\Lambda}_h^0\boldsymbol{w}, \boldsymbol{w})$$

are uniformly equivalent on $\mathbf{V}_h^0$. We have

$$\Lambda_h(\boldsymbol{w}, \boldsymbol{w}) = ((\boldsymbol{I} - \boldsymbol{P}_h)\,\boldsymbol{w} + (\boldsymbol{I} - \mathbf{\Pi}_h)\,\boldsymbol{P}_h\boldsymbol{w}, \mathbf{\Lambda}_h^0\boldsymbol{w}) + (\boldsymbol{P}_h\boldsymbol{w}, \mathbf{\Pi}_h^*\mathbf{\Lambda}_h^0\boldsymbol{w})$$

The first inner product on the right hand side is estimated by

$$(||(\boldsymbol{I} - \boldsymbol{P}_h)\,\boldsymbol{w}\,||_0 + ||(\boldsymbol{I} - \mathbf{\Pi}_h)\,\boldsymbol{P}_h\boldsymbol{w}\,||_0)||\,\mathbf{\Lambda}_h^0\boldsymbol{w}\,||_0 \le ch\Lambda_h(\boldsymbol{w}, \boldsymbol{w})^{1/2}||\,\mathbf{\Lambda}_h^0\boldsymbol{w}\,||_0$$

where we have used the properties (5.10)–(5.12). By the Cauchy–Schwarz inequality, (5.11) and the assumption on $\mathbf{\Theta}_h$ we also have

$$\begin{aligned}(\boldsymbol{P}_h\boldsymbol{w}, \mathbf{\Pi}_h^*\mathbf{\Lambda}_h^0\boldsymbol{w}) &\le \Lambda(\boldsymbol{P}_h\boldsymbol{w}, \boldsymbol{P}_h\boldsymbol{w})^{1/2}(\mathbf{\Lambda}_h^{-1}\mathbf{\Pi}_h^*\mathbf{\Lambda}_h^0\boldsymbol{w}, \mathbf{\Pi}_h^*\mathbf{\Lambda}_h^0\boldsymbol{w})^{1/2} \\ &\le c\Lambda_h(\boldsymbol{w}, \boldsymbol{w})^{1/2}\Lambda_h(\mathbf{\Pi}_h\mathbf{\Theta}_h\mathbf{\Pi}_h^*\mathbf{\Lambda}_h^0\boldsymbol{w}, \boldsymbol{w})^{1/2}\end{aligned}$$

However these bounds imply that

$$\Lambda_h(\boldsymbol{w}, \boldsymbol{w}) \le c\Lambda_h(\mathbf{\Theta}_h^0\mathbf{\Lambda}_h^0\boldsymbol{w}, \boldsymbol{w}) \qquad \text{for all } \boldsymbol{w} \in \mathbf{V}_h^0$$

From (5.2) it follows that the spectral radius of $\mathbf{\Lambda}_h^0$ is $\mathscr{O}(h^{-2})$. Therefore,

$$\Lambda_h(h^2\,\mathbf{\Lambda}_h^0\boldsymbol{w}, \boldsymbol{w}) = h^2||\,\mathbf{\Lambda}_h^0\boldsymbol{w}\,||_0^2 \le c\Lambda_h(\boldsymbol{w}, \boldsymbol{w}) \tag{5.13}$$

This is the first part of the desired lower bound for $\Lambda_h(\boldsymbol{w}, \boldsymbol{w})$. In order to complete the argument note that (5.9) and the Cauchy–Schwarz inequality imply that

$$\begin{aligned}\Lambda_h((\boldsymbol{\Pi}_h\boldsymbol{\Lambda}_h^{-1}\boldsymbol{\Pi}_h^*)\ \boldsymbol{\Lambda}_h^0\boldsymbol{w}, \boldsymbol{w}) &\leq \Lambda_h((\boldsymbol{\Pi}_h\boldsymbol{\Lambda}_h^{-1}\boldsymbol{\Pi}_h^*)\ \boldsymbol{\Lambda}_h^0\boldsymbol{w}, (\boldsymbol{\Pi}_h\boldsymbol{\Lambda}_h^{-1}\boldsymbol{\Pi}_h^*)\ \boldsymbol{\Lambda}_h^0\boldsymbol{w})^{1/2}\Lambda_h(\boldsymbol{w}, \boldsymbol{w})^{1/2}\\ &\leq c\Lambda(\boldsymbol{\Lambda}_h^{-1}\boldsymbol{\Pi}_h^*\boldsymbol{\Lambda}_h^0\boldsymbol{w}, \boldsymbol{\Lambda}_h^{-1}\boldsymbol{\Pi}_h^*\boldsymbol{\Lambda}_h^0\boldsymbol{w})^{1/2}\Lambda_h(\boldsymbol{w}, \boldsymbol{w})^{1/2}\\ &= c\Lambda_h((\boldsymbol{\Pi}_h\boldsymbol{\Lambda}_h^{-1}\boldsymbol{\Pi}_h^*)\ \boldsymbol{\Lambda}_h^0\boldsymbol{w}, \boldsymbol{w})^{1/2}\Lambda_h(\boldsymbol{w}, \boldsymbol{w})^{1/2}\end{aligned}$$

Together with (5.13) and the assumptions on $\boldsymbol{\Theta}_h$ this implies the desired lower bound

$$\Lambda_h(\boldsymbol{\Theta}_h^0\boldsymbol{\Lambda}_h^0\boldsymbol{w}, \boldsymbol{w}) \leq c\Lambda_h(\boldsymbol{w}, \boldsymbol{w}) \qquad \text{for all } \ \boldsymbol{w} \in \mathbf{V}_h^0$$

and this completes the proof of the theorem. ■

5.3. *Proof of Lemma 5.2*

In order to complete the analysis of the auxiliary space preconditioner (5.8) we have to establish the properties (5.9)–(5.12) for the operators $\boldsymbol{\Pi}_h$ and $\boldsymbol{P}_h$.

We first note that (5.9) holds with $c = 1$. The follows from the identity

$$(\text{div } \boldsymbol{v}, q) = -(\boldsymbol{v}, \mathbf{grad}_h q) = (\text{div}_h(\boldsymbol{\Pi}_h \boldsymbol{v}), q) \qquad \text{for all } \ \boldsymbol{v} \in \mathbf{V}_h, q \in W_h^0 \tag{5.14}$$

To see this identity note that for $\boldsymbol{v} \in \mathbf{V}_h$, $(\boldsymbol{v} \cdot \boldsymbol{n})|_E$ is constant on the edges. This implies that

$$\int_E (\boldsymbol{v} \cdot \boldsymbol{n})[q]\,\mathrm{d}s = |E|(\boldsymbol{v} \cdot \boldsymbol{n})(x_E)[q](x_E) = 0 \qquad \text{for all } \ \boldsymbol{v} \in \mathbf{V}_h, q \in W_h^0$$

where x_E is the midpoint of E. This leads directly to (5.14).

Property (5.10) is straightforward. Since the L^2–projection onto $\mathbf{V}_h^0$ is local, we have (letting $\boldsymbol{v} = (v_1, v_2)$),

$$||(\boldsymbol{I} - \boldsymbol{\Pi}_h)\ \boldsymbol{v}\,||_0 \leq ch(\sum_{T\in\mathcal{T}_h} ||\mathbf{grad}\ v_1||_{0,T}^2 + ||\mathbf{grad}\ v_2||_{0,T}^2)^{1/2}$$

However, if $\boldsymbol{v} \in \mathbf{V}_h$ then

$$||\mathbf{grad}\ v_1||_{0,T}^2 + ||\mathbf{grad}\ v_2||_{0,T}^2 = \frac{1}{2}||\text{div }\boldsymbol{v}\,||_{0,T}^2$$

Therefore, (5.10) is established.

We next verify (5.12). Let $\mathbf{V}_h^1$ be the discontinuous Raviart–Thomas space, i.e. $\mathbf{V}_h^1$ has the same degrees of freedom as $\mathbf{V}_h^0$ on each triangle, but the continuity requirements have been removed. Hence, $\mathbf{V}_h, \mathbf{V}_h^0 \subset \mathbf{V}_h^1$. If $\mathbf{z} \in \mathbf{V}_h^1$ then the forms

$$||\,\boldsymbol{z}\,||_0^2 \qquad \text{and} \qquad h^2 \sum_{E\in\mathcal{E}_h} \left((\mathbf{z}\cdot\boldsymbol{n})_{E-}^2 + (\mathbf{z}\cdot\boldsymbol{n})_{E+}^2\right) \tag{5.15}$$

are uniformly equivalent with respect to h. From the definiton of the operator $\boldsymbol{P}_h$ we therefore obtain

$$||(\boldsymbol{I}-\boldsymbol{P}_h)\,\boldsymbol{w}\,||_0^2 \le ch^2\left(\sum_{E\in\mathscr{E}_h}[\boldsymbol{w}\cdot\boldsymbol{n}]_E^2\right) \le ch^2||\mathrm{div}_h\,\boldsymbol{w}\,||_0^2$$

which is (5.12).

Finally, we show that (5.11). It follows directly from (5.15) that $\boldsymbol{P}_h$ is uniformly bounded in L^2. In order to bound $\|\mathrm{div}\,\boldsymbol{P}_h\boldsymbol{w}\,\|_0$ for $\boldsymbol{w}\in\mathbf{V}_h^0$ we note that it follows from a standard inverse inequality that

$$\begin{aligned}\|\mathrm{div}\,\boldsymbol{P}_h\boldsymbol{w}\,\|_0^2 &= \sum_{T\in\mathscr{T}_h}\|\mathrm{div}(\boldsymbol{P}_h\boldsymbol{w}-\boldsymbol{w})\|_{0,\,T}^2\\ &\le ch^{-2}\sum_{T\in\mathscr{T}_h}\|(\boldsymbol{P}_h\boldsymbol{w}-\boldsymbol{w})\|_{0,\,T}^2 \le ch^{-2}\|\,\boldsymbol{P}_h\boldsymbol{w}-\boldsymbol{w}\,\|_0^2.\end{aligned}$$

Here, we have used the fact that $\boldsymbol{w}$ is a constant vector on each triangle. However, together with (5.12) this implies the desired estimate (5.11).

6. Numerical experiments

In this section we shall report on some numerical experiments using the various preconditioners discussed above. The extended domain, Ω_e, will always be taken to be the unit square. The domain Ω is either equal to Ω_e, i.e. there is no effect of domain embedding, or Ω is equal to the L–shaped domain obtained from Ω_e by removing the upper right $1/2\times 1/2$ square. The triangulation of Ω_e is constructed by dividing the unit square into $h\times h$ sized squares, and then dividing each square into two triangles by using the negative sloped diagonal. In all the examples below $\mathbf{V}_h$ and $\mathbf{V}_h\ (\Omega_e)$ will be the lowest order Raviart–Thomas spaces.

6.1. Example

We first consider preconditioners for the operator $\boldsymbol{\Lambda}_h$:$\mathbf{V}_h \mapsto \mathbf{V}_h$. Hence, we consider approximations of the boundary value problem (3.2)–(3.3). In the experiment we have taken $\boldsymbol{f}=(1,\,1)^{\mathrm{T}}$. We investigate the behavior of the preconditioner $\boldsymbol{\Theta}_h$ defined by (4.14), i.e.,

$$\boldsymbol{\Theta}_h=\boldsymbol{R}_h\boldsymbol{\Theta}_{e,h}\boldsymbol{E}_h$$

The operator $\boldsymbol{\Theta}_{e,h}$ is a multigrid V–cycle operator with an additive smoother of the form described in [3], where the scaling factor, η, is taken to be $1/2$.

The condition numbers $\kappa(\boldsymbol{\Theta}_h\boldsymbol{\Lambda}_h)$ are estimated from the conjugate gradient iterations. The results are given in Table 1. Of course, the first column of these results just confirms the theory developed in [3], while the second column seems to agree with the result of Theorem 4.1, i.e., the condition numbers $\kappa(\boldsymbol{\Theta}_h\boldsymbol{\Lambda}_h)$ appear to be bounded uniformly in h.

6.2. Example

In the next example we consider the mixed method for the problem (1.1), with the coefficient a equal to the identity, $f\equiv 1$ and $g\equiv 0$. The discrete system (2.4) is preconditioned by an

Table 1. Condition numbers for the preconditioned $\boldsymbol{H}(\mathrm{div})$–operator

Domain $1/h$	Unit square $\kappa(\boldsymbol{\Theta}_h\boldsymbol{\Lambda}_h)$	L–shaped domain $\kappa(\boldsymbol{\Theta}_h\boldsymbol{\Lambda}_h)$
32	2.40	5.36
64	2.44	5.66
128	2.46	5.90
256	2.46	6.09

Table 2. Condition numbers and the number of iterations for the mixed operator

Domain: $1/h$	Unit square		L–shaped domain	
	$\kappa(\mathcal{B}_h\mathcal{A}_h)$	MINRES	$\kappa(\mathcal{B}_h\mathcal{A}_h)$	MINRES
32	2.33	14	5.31	21
64	2.35	14	5.62	21
128	2.34	14	5.86	22
256	2.34	14	6.05	22

operator $\mathcal{B}_h$ of the form 3.4, i.e.,

$$\mathcal{B}_h = \begin{pmatrix} \boldsymbol{\Theta}_h & 0 \\ 0 & I \end{pmatrix} : \mathbf{V}_h \times W_h \mapsto \mathbf{V}_h \times W_h$$

Here W_h is the space of piecewise constants and the $H(\mathrm{div})$–preconditioner $\boldsymbol{\Theta}_h$ is chosen exactly as in the previous example above. In Table 2 we present the results of this experiment. In addition to the estimates for the condition numbers, $\kappa(\mathcal{B}_h\mathcal{A}_h)$, we also report the number of iterations required by the minimum residual method to reduce the residual of the preconditioned system by a factor 10^{-5} in the norm induced by the inner product $(\mathcal{B}_h^{-1}\cdot, \cdot)$. As expected, the results appear to be bounded, independently of h.

6.3. Example

Finally, we consider the nonconforming method studied in Section 5, i.e., the system (5.1). This system is formulated as a saddle point problem, cf. (5.3), and is preconditioned by a block diagonal operator

$$\mathcal{B}_h = \begin{pmatrix} \boldsymbol{\Theta}_h^0 & 0 \\ 0 & I_h \end{pmatrix} : \mathbf{V}_h^0 \times W_h^0 \mapsto \mathbf{V}_h^0 \times W_h^0$$

We recall that $\mathbf{V}_h^0$ is the space of piecewise constant vectors, while W_h^0 is the piecewise linear Crouzeix–Raviart space. Here the preconditioner $\boldsymbol{\Theta}_h^0{:}\mathbf{V}_h^0 \mapsto \mathbf{V}_h^0$ is the auxiliary space preconditioner given by (5.8), i.e.,

$$\boldsymbol{\Theta}_h^0 = \tau h^2 \boldsymbol{I} + \boldsymbol{\Pi}_h \boldsymbol{\Theta}_h \boldsymbol{\Pi}_h^*$$

Table 3. Condition numbers and the number of iterations for the non-conforming operator

Domain: $1/h$	Unit square		L–shaped domain	
	$\kappa(\mathcal{B}_h\mathcal{A}_h)$	MINRES	$\kappa(\mathcal{B}_h\mathcal{A}_h)$	MINRES
32	2.79	25	5.30	32
64	2.79	24	5.61	34
128	2.79	24	5.86	34
256	2.78	22	5.85	34

In the experiments we have chosen $\tau = 0.01$ and $\mathbf{\Theta}_h:\mathbf{V}_h \mapsto \mathbf{V}_h$ exactly as in the two previous examples. The operator I_h on W_h^0 is obtained from the identity operator by replacing exact integration by the simplest numerical integration rule based on the values at the midpoint of each edge. In the same way as the above the iterations are terminated when the proper residual is reduced by a factor of 10^{-5}. The results are given in Table 3. Since the condition numbers appear to be bounded, independently of h, this confirms, indirectly, the conclusion of Theorem 5.1

If the parameter τ is increased from 0.01 to 0.05 the condition numbers seem to increase by a factor of at most 3/2, and usually much less. Hence, the performance of the iterative solvers is not too sensitive with respect to perturbations in τ. Furthermore, it seems from the experiments that a suitable chice of τ can be made independent of the mesh parameter h.

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